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THE ASYMPTOTIC THERMAL STABILITY OF CONFINED FLUIDS*

by

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SUMMARY

This paper investigates the time dependent thermal stability of completely confined fluids.

The physical model is a fluid enclosed in a rigid container of arbitrary shape. Part of the container walls are heated and the remainder is insulated. The resulting flow field and its dependence on the time are the object of the research.

Mathematically the problem is an initial-boundary value problem and the main tool for its treatment is functional analysis.

The following results are obtained:

a. There exists no slow time-independent flow field except the rest state.
b. A rest state is reached if $\|K_\ell\|_{Z_{Pr}}<1$, K_ℓ is the characteristic operator of the problem and Z_{Pr} is the Hilbert space in which the problem is defined.
c. With the addition of restriction on the body force it is shown that the rest state can exist only if the condition in b is satisfied.

1. *Introduction*

The time dependent thermal stability of completely confined fluids is a particular case of natural convection in closed containers.

A fluid is completely confined in a container which is heated from the outside. A density gradient results from the non-uniform temperature distribution and the body forces may induce a flow; i.e. the temperature and the flow fields are related. The natural convection is characterized by this interrelation between the internal flow and the temperature distribution within the fluid.

To make the problem mathematically tractable the following assumptions were made:

1. The fluid is Newtonian
2. The flow is laminar
3. Fluid properties do not depend on temperature
4. The fluid is mechanically incompressible
5. The density gradient is small
6. The increase in the internal energy due to the work done by the viscous forces is small compared to changes in the internal energy caused by heat transfer.

Since the classical theory of Navier-Stokes is based on the first three assumptions, their domain of applicability is well known. The fourth assumption is generally valid for fluids since the density changes are small over a wide range of pressures. The density gradient is considered "small"

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when some characteristic temperature gradient imposed on the fluid by the boundary conditions is small compared to the ratio $1/\alpha h$, where h is the "height" of the container and α the coefficient of thermal expansion. A fluid which satisfies both assumptions 4) and 5) is sometimes called "quasi-incompressible". The last assumption which neglects the dissipation in the energy equation is valid for slow flows with high rate of heat transfer.

Under these assumptions, the natural convection is described by: [1:§56]

$$\begin{cases} \nabla \cdot \underline{q} = 0 \\ \frac{\partial}{\partial t}\underline{q} + (\underline{q}\cdot\nabla)\underline{q} = -\frac{1}{\rho}\nabla P + \nu\Delta\underline{q} - \alpha T \underline{G} \\ \frac{\partial}{\partial t} T + \underline{q}\cdot\nabla T = k\Delta T \end{cases} \tag{1}$$

where
Δ - Laplace operator
k - coefficient of thermal diffusion
ν - kinematic viscosity
$\underline{q}$ - velocity vector
T - temperature
P - pressure
$\underline{G}$ - body force field
ρ - density

The heating conditions are such that they admit, at least asymptotically, a zero flow solution to Eq. 1).

The stability of the fluid depends on whether such a rest state can or cannot be reached. This is shown to be a function of some critical values of the governing parameters. The object of this work is to consider these parameters and show how they influence thermal stability.

2. The Statement of the Problem

A container, R, of arbitrary shape and rigid walls, ∂R, is completely filled with fluid. The container is heated from the outside such that the temperature $\mathcal{T}$, is given on part of its walls, $\partial R'$, and the remainder part, $\partial R''$, ($\partial R = \partial R' + \partial R''$, $\partial R' \neq 0$) is heated with a known rate, Q. Both the functions $\mathcal{T}$ and Q approach asymptotically values $\mathcal{T}_\infty$ and Q_∞, respectively, as $t \to \infty$. The functions $\mathcal{T}_\infty$ and Q_∞ are independent of time but, of course, need not have the same values everywhere.

In the absence of a body force, the temperature field in the fluid can be obtained from the Fourier equation:

$$\begin{cases} \frac{\partial \overset{o}{T}}{\partial t} - k\Delta\overset{o}{T} = 0 \\ t \leqslant 0: \text{ given } \overset{o}{T} \\ t > 0: \left.\overset{o}{T}\right|_{\partial R'} = \mathcal{T} \; ; \; k\left.\frac{\partial T}{\partial n}\right|_{\partial R''} = Q \end{cases}$$

Since $\partial R'$ was assumed to be different from zero, the field $\overset{o}{T}$ approaches asymptotically a steady state, $\overset{o}{T}_\infty$ which satisfies:

$$\begin{cases} \Delta\overset{o}{T}_\infty = 0 \\ \left.\overset{o}{T}_\infty\right|_{\partial R'} = \mathcal{T}_\infty \; ; \; k\left.\frac{\partial \overset{o}{T}_\infty}{\partial n}\right|_{\partial R''} = Q_\infty \end{cases}$$

Let $\underline{G}$ be a time dependent body force acting on the fluid, and let $\underline{G}$ be conservative in the sense that it is a gradient of some potential. In this case the fluid cannot remain at rest unless the body force satisfies:

$$\underline{G} \times \nabla \overset{o}{T} = 0 \tag{2}$$

When the above condition is not satisfied there is no adequate hydrostatic pressure [2 : §1]. The condition (2) is not sufficient for the fluid to be in the rest state since it still may be unstable. In previous work the problem was reduced to the investigation of the stability of the rest state with no considerations as to how this rest state is reached (if it can be reached at all).

An internal flow is likely to start at the beginning of the heating. Moreover, the body force $\underline{G}$ satisfies the condition (2) only in an asymptotical manner; i. e,

$$\underline{G}_{\infty} \times \nabla \overset{o}{T}_{\infty} = 0 \tag{3}$$

therefore, the asymptotical behaviour of the flow, rather than the stability of the rest state, has to be investigated. This work investigates this asymptotical behaviour.

The results obtained are:

1. The internal flow dissipates out and the rest state is asymptotically reached if a certain relation between $\underline{G}_{\infty}$ and $\nabla \overset{o}{T}_{\infty}$ - the criterion of the stability - is satisfied, regardless of the history of the asymptotic fields $\underline{G}_{\infty}$ and $\nabla \overset{o}{T}_{\infty}$.
2. a) When this relation does not hold, the internal flow cannot attain any small, time-independent asymptotical value except, possibly, the rest state, It is not shown that the rest state cannot occur.

   b) For $\underline{G}$ restricted to

$$\underline{G}_{\infty} = \beta \nabla \overset{o}{T}_{\infty}, \; \beta = \text{constant} \tag{4}$$

   it is shown that the rest state cannot be reached unless the stability criterion holds.

   The asymptotic boundary conditions are time-independent; furthermore the stability criterion may be made not to hold by the addition of arbitrarily small ϵ to one side of the relation, yet, when $\underline{G}_{\infty} = \beta \nabla \overset{o}{T}_{\infty}$, the asymptotical values of the flow field are either time-dependent, or must be large. There is some indirect evidence [4] that the asymptotical flow is time-dependent. Still the results for that case (the stability criterion does not hold) leave much room for further investigation.

3. *The Basic Equations*

Let the basic equations (1) be made non-dimensional by the use of the following characteristic values:

time	temperature	acceleration	velocity	pressure
$\frac{h^2}{\sqrt{\nu k}}$	$\frac{1}{\alpha}$	$\frac{\nu k}{h^3}$	$\frac{\sqrt{\nu k}}{h}$	$\frac{\rho \nu k}{h^2}$

Fig. 1. Characteristic values

Further, let $T = \overset{o}{T} + \theta$ and $P = \overset{o}{P}_\infty + p$ *) be introduced into the basic equations which thus become:

$$\begin{cases} \nabla \cdot \underline{q} = 0 \\ \frac{D}{Dt}\underline{q} - Pr^{1/2}\Delta\underline{q} + \theta\underline{G} = \nabla p - (\overset{o}{T}\underline{G} - \overset{o}{T}_\infty \underline{G}_\infty) \\ \frac{D}{Dt}\theta - Pr^{-1/2}\Delta\theta + \underline{q}\cdot\nabla\overset{o}{T} = 0 \\ t \leqslant 0 : \text{ given } \theta \text{ and } q \\ t > 0 : \underline{q}\Big|_{\partial R} = 0; \ \theta\Big|_{\partial R'} = 0; \ \frac{\partial\theta}{\partial n}\Big|_{\partial R''} = 0. \end{cases} \qquad (5)$$

These equations contain the Prandtl Number, $Pr = \frac{\nu}{k}$, as a parameter, and two non-dimensional functions: the body force field $\underline{G}$, and the temperature field, $\overset{o}{T}$, of the hypothetical rest state; these are given or computed beforehand.

When the solutions of the basic equations approach time-independent steady-state they must satisfy:

$$\begin{cases} \nabla\cdot\underline{q}_\infty = 0 \\ (\underline{q}_\infty\cdot\nabla)\underline{q}_\infty - Pr^{1/2}\Delta\underline{q}_\infty + \theta_\infty\underline{G}_\infty = \nabla p_\infty \\ (\underline{q}_\infty\cdot\nabla)\theta_\infty - Pr^{-1/2}\Delta\theta_\infty + \underline{q}\cdot\nabla\overset{o}{T}_\infty = 0 \\ \underline{q}_\infty\Big|_{\partial R} = 0 \ ; \ \theta_\infty\Big|_{\partial R'} = 0 \ ; \ \frac{\partial\theta_\infty}{\partial n}\Big|_{\partial R''} = 0 \end{cases} \qquad (6)^{**}$$

and all the other terms vanish asymptotically

4. *Some Particular Integral Equalities*

The following integral equalities are derived from the basic equations (5): (the summation convention is adopted everywhere)

$$\frac{1}{2}\frac{d}{dt}\int_R q_i q_i \, dV = -Pr^{1/2}\int_R \frac{\partial q_i}{\partial x_j}\frac{\partial q_i}{\partial x_j}dV - \int_R \theta G_i q_i dV - \int_R (\overset{o}{T}G_i - \overset{o}{T}_\infty G_{i\infty})q_i dV$$

$$\frac{1}{2}\frac{d}{dt}\int_R \theta^2 dV = -Pr^{-1/2}\int_R \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_i}dV - \int_R \theta q_i \frac{\partial\overset{o}{T}_\infty}{\partial x_i}dV \qquad (7)$$

Proof: consider the scalar product of the momentum equations and $\underline{q}$,

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*) The existence of such a "hydrostatic pressure" which satisfies

$$\nabla\overset{o}{P}_\infty = \rho\overset{o}{T}_\infty\underline{G}_\infty$$

is guarranteed by the condition (3).

**) Note that if the body force is restricted to satisfy (4), then the basic equations (6) can be further simplified by a new change of variables:

$$\underline{G}^* = \frac{1}{\sqrt{\beta}}\underline{G} \ ; \ \overset{o}{T}^* = \sqrt{\beta}\overset{o}{T} \ ; \ \theta^* = \sqrt{\beta}\theta$$

where $\frac{\alpha k\nu}{h^3}$ was used as scale to β. With this change of variables the basic equations look the same as Eq.(6) but $\underline{G}_\infty$ equals now $\nabla\overset{o}{T}_\infty$.

Restriction (4) is required in the proof of result 2 b).

and the product of the energy equation and θ. Integration over the whole region R, the use of Green's theorem and the boundary conditions lead to Eq.(7). The following equalities are intermediate steps.

1)

$$\int_R q_i \Delta q_i dV = \int_R q_i \frac{\partial^2 q_i}{\partial x_j \partial x_j} dV = \int_{\partial R} q_i \frac{\partial q_i}{\partial x_j} n_j dS + - \int_R \frac{\partial q_i}{\partial x_j} \frac{\partial q_i}{\partial x_j} dV$$

Because $\underline{q}\big|_{\partial R} = 0$ this leads to:

$$\int_R \underline{q} \cdot \Delta \underline{q} dV = - \int_R \frac{\partial q_i}{\partial x_j} \frac{\partial q_i}{\partial x_j} dV$$

2)

$$\int_R \theta \Delta \theta dV = \int_R \theta \frac{\partial^2 \theta}{\partial x_i \partial x_i} dV = \int_{\partial R} \theta \frac{\partial \theta}{\partial x_i} n_i dS + - \int_R \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_i} dV$$

Because $\theta \big|_{\partial R'} = 0,\ \frac{\partial \theta}{\partial x_i} n_i \big|_{\partial R'} = 0$ and $\partial R = \partial R' + \partial R''$ this leads to:

$$\int_R \theta \Delta \theta dV = - \int_R \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_i} dV$$

3)

$$\int_R \left[(\underline{q} \cdot \nabla)\underline{q}\right] \cdot \underline{q} dV = \int_R q_i \frac{\partial q_j}{\partial x_i} q_j dV = \frac{1}{2} \int_R q_i \frac{\partial (q_j q_j)}{\partial x_i} dV$$

$$= \frac{1}{2} \int_{\partial R} q_j q_j q_i n_i dS - \frac{1}{2} \int_R q_i q_j \frac{\partial q_i}{\partial x_i} dV$$

Because $\nabla \cdot \underline{q} = \frac{\partial q_i}{\partial x_i} = 0$ and $\underline{q}\big|_{\partial R} = 0$ this leads to:

$$\int_R \left[(\underline{q} \cdot \nabla)\underline{q}\right] \cdot \underline{q} \, dV = 0$$

4)

$$\int_R \theta \underline{q} \cdot \nabla \theta \, dV = \int_R \theta q_i \frac{\partial \theta}{\partial q_i} dV = \frac{1}{2} \int_R q_i \frac{\partial \theta^2}{\partial x_i} dV$$

$$= \frac{1}{2} \int_{\partial R} \theta^2 q_i n_i dS - \int_R \theta^2 \frac{\partial q_i}{\partial x_i} dV$$

Because $\underline{q}\big|_{\partial R} = 0$ and $\nabla \cdot \underline{q} = 0$ this leads to:

$$\int_R \theta \underline{q} \cdot \nabla \theta \, dV = 0$$

5)

$$\int_R \underline{q} \frac{\partial}{\partial t} \underline{q} \, dV \equiv \int_R q_i \frac{\partial}{\partial t} q_i \, dV = \frac{1}{2} \frac{d}{dt} \int_R q_i q_i \, dV$$

6)

$$\int_R \theta \frac{\partial}{\partial t} \theta \, dV = \frac{1}{2} \frac{d}{dt} \int_R \theta^2 dV$$

The addition of the two equalities (7) yields:

$$\frac{1}{2} \frac{d}{dt} \int_R (\theta^2 + q_i q_i) dV = \Bigg[\int_R \Big(Pr^{-1/2} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_i} + Pr^{1/2} \frac{\partial q_i}{\partial x_j} \frac{\partial q_i}{\partial x_j} \Big) dV + \int_R \theta \Big(G_{i\infty} + \frac{\partial \mathring{T}_\infty}{\partial x_i} \Big) q_i dV \Bigg] + - \int_R \theta \Big[(G_i - G_{i\infty}) + \frac{\partial}{\partial x_i} (\mathring{T} - \mathring{T}_\infty) \Big] q_i dV + - \int_R (\mathring{T} G_i - \mathring{T}_\infty G_{i\infty}) q_i dV \tag{8}$$

If the field $\underline{G}$ satisfies the additional restriction (4) then the solutions θ and $\underline{q}$ must also satisfy:

$$\int_R \left[\left(\frac{\partial \theta}{\partial t} \right)^2 + \frac{\partial q_i}{\partial t} \frac{\partial q_i}{\partial t} \right] dV = - \int_R \frac{\partial q_i}{\partial x_j} q_j \frac{\partial q_i}{\partial t} dV - \int_R \frac{\partial \theta}{\partial t} q_i \frac{\partial \theta}{\partial x_i} dV + \frac{1}{2} \frac{d}{dt} \Bigg[\int_R \Big(Pr^{-1/2} \frac{\partial \theta}{\partial x_j} \frac{\partial \theta}{\partial x_j} + Pr^{1/2} \frac{\partial q_i}{\partial x_j} \frac{\partial q_i}{\partial x_j} \Big) dV + \int_R \theta \Big(G_{i\infty} + \frac{\partial \mathring{T}_\infty}{\partial x_i} \Big) q_i dV + - \int_R \theta (G_i - G_{i\infty}) \frac{\partial q_i}{\partial T} dV - \int_R \frac{\partial \theta}{\partial t} q_i \frac{\partial}{\partial x_i} (\mathring{T} - \mathring{T}_\infty) dV - \int_R (\mathring{T} G_i - \mathring{T}_\infty G_{i\infty}) \frac{\partial q_i}{\partial t} dV \tag{9}$$

Proof: Consider the scalar product of the momentum equations and $\frac{\partial}{\partial t}\underline{q}$. and the product of the energy equation and $\frac{\partial \theta}{\partial t}$. Integration over the whole region R, the use of Green's theorem and boundary conditions lead to Eq. (9) Some intermediare steps are:

1)

$$\int_R \frac{\partial \underline{q}}{\partial t} \cdot \Delta \underline{q} \, dV = \int_R \frac{\partial q_i}{\partial t} \frac{\partial^2 q_i}{\partial x_j \partial x_j} dV = - \int_R \frac{\partial^2 q_i}{\partial x_j \partial t} \frac{\partial q_i}{\partial x_j} dV =$$

$$= - \int_R \frac{\partial q_i}{\partial x_j} \frac{\partial}{\partial t} \left(\frac{\partial q_i}{\partial x_j} \right) dV = - \frac{1}{2} \frac{d}{dt} \int_R \frac{\partial q_i}{\partial x_j} \frac{\partial q_i}{\partial x_j} dV$$

$$\int_R \frac{\partial \underline{q}}{\partial t} \cdot \Delta \underline{q}\, dV = -\frac{1}{2}\frac{d}{dt}\int_R \frac{\partial q_i}{\partial x_j}\frac{\partial q_i}{\partial x_j}\, dV$$

Likewise

$$\int_R \frac{\partial \theta}{\partial t}\, \Delta \theta\, dV = -\frac{1}{2}\frac{d}{dt}\int_R \frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_i}\, dV$$

2)

Because $\underline{G}$ satisfies the restriction 4)

$$\int \left(\theta G_{i\infty}\frac{\partial q}{\partial t} + \frac{\partial \theta}{\partial t}\frac{\partial \overset{\circ}{T}_\infty}{\partial x_i} q_i\right) dV = \frac{1}{2}\frac{\partial}{\partial t}\int_R \theta\left(G_{i\infty} + \frac{\partial \overset{\circ}{T}_\infty}{\partial x_i}\right) q_i\, dV$$

(see the note on pg. 56)

Half of the time derivative of equality (8) is now substracted from Eq. (9) to yield:

$$\frac{1}{4}\frac{d^2}{dt^2}\int_R (\theta^2 + q_i q_i) dV = \int_R \left[\left(\frac{\partial \theta}{\partial t}\right)^2 + \frac{\partial q_i}{\partial t}\cdot\frac{\partial q_i}{\partial t}\right] dV + \int_R \frac{\partial q_i}{\partial t} q_j \frac{\partial q_i}{\partial x_j} dV + \int_R \frac{\partial \theta}{\partial t} q_i \frac{\partial \theta}{\partial x_i} dV$$

$$-\frac{1}{2}\frac{d}{dt}\left\{\int_R \theta\left[(G_i - G_{i\infty}) + \frac{\partial}{\partial x_i}(\overset{\circ}{T} - \overset{\circ}{T}_\infty)\right] q_i dV - \int_R (\overset{\circ}{T} G_i - \overset{\circ}{T}_\infty G_{i\infty}) q_i dV\right\} +$$

$$+\int_R \theta (G_i - G_{i\infty})\frac{\partial q_i}{\partial t} dV + \int \frac{\partial \theta}{\partial t} q_i \frac{\partial}{\partial x_i}(\overset{\circ}{T} - \overset{\circ}{T}_\infty) dV + \int_R (\overset{\circ}{T} G_i - \overset{\circ}{T}_\infty G_{i\infty})\frac{\partial q_i}{\partial x_i} dV \qquad (10)$$

5. *Some General Inequalities*

1. Let (C.B.S.) denote the Cauchy-Buniakowsky-Schwarz-inequality [8 : §1]

$$\left|\int_R f_1 f_2\, dV\right| \leqq \left(\int_R f_1^2\, dV\right)^{1/2}\left(\int_R f_2^2\, dV\right)^{1/2}$$

(the existence of the integrals on the right hand side is assumed in this section)

2. Let (H) denote the Hölder-inequality [10 : §1]

$$\left|\int_R (f_1 f_2 \dots f_n)\, dV\right| \leqslant \left(\int_R f_1^{p_1}\, dV\right)^{1/p_1}\left(\int_R f_2^{p_2}\, dV\right)^{1/p_2} \dots \left(\int_R f_n^{p_n}\, dV\right)^{1/p_n}$$

where p_1, $p_2, \dots, p_n$ are all positive and satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \dots \frac{1}{p_n} = 1$$

3. Let (L) denote the inequality

$$\int_R f_i^2 f_i^2 dV \leq 4 \left(\int_R f_i f_i dV\right)^{1/2} \left(\int_R \frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_j} dV\right)^{3/2}$$

4. Let (F) denote the inequality

$$\int_R f^2 dV \leq C^2 \int_R \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} dV$$

where f vanishes on $\partial R'$ and has a vanishing normal derivative on $\partial R''$. The (F) inequality is a consequence of the existence of the lowest eigen-value λ_1, of the Helmholtz equation with mixed boundary conditions [15 : XXV §3]:

$$\begin{cases} \Delta f + \lambda^2 f = 0 \\ \left(\alpha \dfrac{f}{n^2} + \beta f\right)\Bigg|_{\partial R} = 0 \end{cases}$$

where α and β are point functions defined on ∂R.

6. *Some Elements of Functional Analysis*

6.1. A real Hilbert Space E is defined as a complete normed real linear space with a scalar product; i.e, a collection of elements x, y, z.... with the following properties:

Linearity a) For any two elements $x, y \in E$ the sum $x + y \in E$ is defined, and $x + y = y + x$; Furthermore for $x, y, z \in E$ $x + (y + z) = x + (y + z)$.

b) For any real numbers λ and μ the element $\lambda x \in E$ is defined for every $x \in E$, and $\lambda(x + y) = \lambda x + \lambda y$; $\lambda(\mu x) = (\lambda\mu)x$; $(\lambda + \mu)x = \lambda x + \mu x$.

c) There exists a unique element Q such that $Q\,x = Q$ and $Q + x = x$ for every $x \in E$.

The norm

d) There exists a real valued non-negative function (called the norm) defined on E and denoted by $\| \ \|$ such that $\|\lambda x\| = |\lambda| . \|x\|$ for every real λ and $x \in E$ (therefore $\|Q\| = 0$), $\|x\| > 0$ if $x \neq Q$ and $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality) for every $x, y \in E$.

Completeness

e) If $x_1, x_2, \ldots\ldots$ is a sequence of elements of E such that

$$\lim_{m,n \to \infty} \|x_n - x_m\|$$

then there exists an element $x_0 \in E$ (necessarily unique) such that $x_u \longrightarrow x_0$; i.e.,

$$\lim_{n \to \infty} \|x_n - x_0\| = 0$$

(as usual x-y means $x + (-1)y \in E$)

The scalar product

f) There exists a real-valued function (called the scalar product) defined on ExE and denoted by < ; > such that $<x;x> = \|x\|^2$ and $<x ; y> = <y ; x>$ for every $x, y \in E$; furthermore, $<\lambda x_1 + \mu x_2 ; y> = \lambda<x_1 ; y) + \mu<x_2 ; y>$ for all real λ, μ and $x, y \in E$. The scalar product also satisfies $<x;y> \leqslant \|x\| \cdot \|y\|$, which results from the previous definitions.

If a space E satisfies all the requirements but (e), it is not complete. It is always possible to adjoin new elements to E and to define for these new elements the algebraic operations, the norm and the scalar product (without altering them for the original elements of E) in such a way that the resulting collection of elements (called the completion of E) satisfies a)-f); furthermore, for any new elements Z there is a sequence $x_1, x_2, \ldots$ in E, which converges to Z.

A set S, $S \subset E$ and E complete, is defined as dense in E if every element z, $z \in E$, is the limit of a sequence $\{x_n \in S\}$.

A Hilbert space is completely defined by any dense set of it and the scalar product (which actually, defines the norm). In any particular situation, the new elements may be of a character quite different from the original elements of E, in the same way, e.g. as the completion of the rational numbers leads to the real number system.

The Hilbert space E_1 is said to be embedded in the space E_2, $E_1 \subset E_2$, if the same set S is dense in both E_1 and E_2, and, in addition, there exists a positive ϵ such that

$$\| \ \|_{E_2} \leqslant \epsilon \| \ \|_{E_1} .$$

Obviously, this means that every element of E_1 is an element of E_2.

If $E_1 \subset E_2$ and also $E_2 \subset E_1$, both the spaces E_1 and E_2 contain only identical elements. The norms in E_1 and E_2 are said to be equivalent [9 : §112].

In the following considerations the elements of the Hilbert spaces will be either scalar or vectorial fields (point functions) defined on R, or ordered pairs of such fields.

The Hilbert space L_2 consists of all functions which are square integrable over R. The linear operations are defined in the usual way (addition of functions and multiplication by numbers).

The scalar product and the norm are

$$\begin{cases} < f_1 ; f_2 > = \int_R f_1 f_2 dV \\ \| f \|_{L_2} = \left(\int_R f^2 dV \right)^{1/2} \end{cases}$$

The set C^∞ of all infinitely differentiable functions is dense in L_2 [14 : § 8].

The Hilbert space $\mathbf{L}_2$ is the vectorial counterpart of L_2. The scalar product and, respectively, the norm are defined by:

$$\begin{cases} <\underline{U} \; ; \; \underline{V}> = \int_R U_i V_i dV \\ \| \underline{U} \|_{L_2} = \left(\int_R U_i U_i dV \right)^{1/2} \end{cases}$$

The Hilbert products space $L_2 \times L_2$; i.e, the space whose elements are ordered pairs (u, $\underline{U}$) with $u \in L_2$ and $\underline{U} \in L_2$, is denoted by $\mathcal{L}$. In $\mathcal{L}$ the scalar product and, respectively, the norm are defined by:

$$\begin{cases} <(u, \underline{U}) \; ; \; (v, \underline{V})> = \int_R (uv + U_i V_i) dV \\ \|(u, \underline{U})\|_{\mathcal{L}} = \left(\int_R (u^2 + U_i U_i) dV \right)^{1/2} \end{cases}$$

The set $\mathfrak{C}^\infty = C^\infty \times \mathbb{C}^\infty$ is dense in $\mathcal{L}$.

Let ϵ be any positive constant and let a new norm be defined

$$\|(u, \underline{U})\|_{\mathcal{L}_\epsilon} = \left[\int_R (\epsilon^{-1/2} u^2 + \epsilon^{1/2} U_i U_i) dV \right]^{1/2}$$

The completion of $\mathfrak{C}^\infty$ in the new norm leads to the Hilbert space denoted by $\mathcal{L}_\epsilon$.

Since this new norm van easily be proved to be equivalent to the other one the spaces $\mathcal{L}$ and $\mathcal{L}_\epsilon$ contain only identical elements.

6.2. Generalized Derivatives in Hilbert Spaces Let $D^\ell \varphi$ denote $\dfrac{\partial^\ell \varphi}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \partial x_3^{\ell_3}}$

$\ell_1 + \ell_2 + \ell_3 = \ell$; the function ψ is called the generalized derivative of the type $D^\ell \varphi$ of a function φ in R, if there exists a sequence of functions φ_m, ℓ times continuously differentable inside R, such that φ_m and $D^\ell \varphi_m$ are convergent to φ and ψ respectively, in any domain R' strictly interior to R; the convergence has to be in the L_2 norm.

Henceforth, all derivatives will be interpreted in the generalized sense. The properties of the generalized derivatives, in particular, the coincidence of the generalized derivative and the usual derivative, when this latter exists, were proved in [9 : §109], [10 : §5].

The Hilbert space W_2^ℓ consists of all functions φ which are measurable on R, have derivatives $D^k \varphi$ of all order $k \leqslant \ell$, and are such that both the function and all these derivatives are square-integrable over R. The scalar product and the norm are:

$$\begin{cases} < \varphi \; ; \; \psi > = \int_R \dfrac{\partial^i \varphi}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} \dfrac{\partial^i \psi}{\partial x_2^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} dV \\ \qquad\qquad i = 0, 1, \ldots, \ell \\ \qquad\qquad i_1 + i_2 + i_3 = i \\ \| \varphi \|_{W_2^\ell} = (< \varphi \; ; \; \varphi >)^{1/2} \end{cases}$$

The set C^∞ is dense in W_2^ℓ. Moreover, if $\varphi_1, \varphi_2 \ldots.$ is the converging sequence of φ_0 then $D^k\varphi_1, D^k\varphi_2, \ldots..$ is the sequence which converges to $D^k\varphi_0$ for all $k \leqslant \ell$ [9 : §112].

The Hilbert space $\mathbf{W}_2^\ell$ is the vectorial counterpart of W_2^ℓ; The scalar product and the norm in $\mathbf{W}_2^\ell$ are:

$$\begin{cases} <\underline{\phi}, \underline{\psi}> = \displaystyle\int_R \frac{\partial^i \phi_j}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} \frac{\partial^i \psi_j}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} \, dV \\ \qquad\qquad i = 0, 1 \ldots, \ell \\ \qquad\qquad i_1 + i_2 + i_3 = i \\ \| \underline{\phi} \|_{\mathbf{W}_2^\ell} = (< \underline{\phi} ; \underline{\phi}>)^{1/2} \end{cases}$$

Let the space $\mathcal{W}^\ell$ denote the product space $W_2^\ell \times \mathbf{W}_2^\ell$. In this Hilbert space the scalar product and the norm are:

$$\begin{cases} <(\varphi, \underline{\phi}) ; (\psi, \underline{\Omega})> = < \varphi ; \omega > + < \underline{\phi} ; \underline{\Omega} > \\ \|(\varphi, \underline{\phi})\|_{\mathcal{W}_1} = (<(\varphi, \underline{\phi}) ; (\varphi, \underline{\phi})>)^{1/2} \end{cases}$$

The set $\mathcal{C}^\infty$ is dense in $\mathcal{W}^\ell$. Moreover, for any $\epsilon > 0$, the completion of $\mathcal{C}^\infty$ in the following norm:

$$\|(\varphi, \underline{\phi})\|_{\mathcal{W}_\epsilon^\ell} = (\epsilon^{-1/2} \|\varphi\|^2_{W_2^\ell} + \epsilon^{1/2} \|\underline{\phi}\|^2_{\mathbf{W}_2^\ell})^{1/2}$$

leads to a Hilbert space $\mathcal{W}_\epsilon^\ell$. Both $\mathcal{W}^\ell$ and $\mathcal{W}_\epsilon^\ell$ have identical elements only, since the two norms are obviously equivalent,

$C^1_{\partial R}$ is defined as the set of all functions which belong to C^1 and vanish on the boundary ∂R $C^1_{\partial R'}$ is defined as the set of all functions which belong to C^1 and satisfy the following boundary conditions:

$$f \Big|_{\partial R'} = 0 \; ; \; \frac{\partial f}{\partial n} \Big|_{\partial R''} = 0$$

Both $C^1_{\partial R}$ and $C^1_{\partial R'}$ are subsets of C^1. $C^\ell_{\partial R}$ is defined as the set of all functions which belong to both C^ℓ and $C^1_{\partial R}$. $C^\ell_{\partial R'}$ is defined as the set of all functions which belong to both C^ℓ and $C^1_{\partial R'}$. The Hilbert spaces $W^\ell_{2, \partial R}$ and $W^\ell_{2, \partial R'}$, obtained from $C^\ell_{\partial R}$ and $C^\ell_{\partial R'}$, respectively by completion in the W_2^ℓ norm, are obviously embedded in the space W_2^ℓ. In these spaces Green's Theorem holds and, therefore, the functions from $W^\ell_{2, \partial R}$ and $W^\ell_{2, \partial R'}$ satisfy homogeneous boundary conditions [9 : §112] in the sense of Green's Theorem; i.e.

$$\int_R \psi \frac{\partial^2 \varphi}{\partial x_i \partial x_i} \, dV = \int_R \varphi \frac{\partial^2 \psi}{\partial x_i \partial x_i} \, dV = - \int_R \frac{\partial \varphi \, \partial \psi}{\partial x_i \partial x_i} \, dV$$

The Hilbert space $\mathbf{W}^\ell_{2, \partial R}$ and $\mathbf{W}^\ell_{2, \partial R'}$ are vectorial counterparts of $W^\ell_{2, \partial R}$ and $W^\ell_{2, \partial R'}$ respectively. Green's Theorem

$$\begin{cases} \displaystyle\int_R \underline{\phi} \nabla \cdot \underline{\psi} \, dV = - \int_R \underline{\psi} \nabla \cdot \underline{\phi} \, dV \\ \displaystyle\int_R \underline{\phi} \cdot \nabla \underline{\psi} \, dV = \int_R \underline{\psi} \cdot \nabla \underline{\phi} \, dV \end{cases}$$

holds in this space

Let $\mathcal{W}_\partial^\ell$ denote the product space $W^\ell_{2,\partial R'} \times \mathbf{W}^\ell_{2,\partial R}$. In this space the following equality is the equivalent of Green's Theorem.

$$\int_R (\varphi\,\Delta\omega + \underline{\phi}\cdot\Delta\underline{\Omega})dV = \int_R \left(\varphi\frac{\partial^2\omega}{\partial x_i\partial x_i} + \phi_i\frac{\partial^2\Omega_i}{\partial x_j\partial x_j}\right)dV =$$

$$= \int_R \left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\omega}{\partial x_i} + \frac{\partial\phi_i}{\partial x_j}\frac{\partial\Omega_i}{\partial x_j}\right)dV =$$

$$= \int_R (\omega\,\Delta\psi + \underline{\Omega}\cdot\Delta\underline{\phi})dV$$

The spaces $D_{\partial R}$ and, respectively, $D_{\partial R'}$ are obtained from the sets $C^1_{\partial R}$ and $C^1_{\partial R'}$ by completion in the following norm

$$\|\varphi\|_D = \left(\int_R \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dV\right)^{1/2}$$

$D_{\partial R}$ and $D_{\partial R'}$ are Hilbert spaces with the scalar product defined by:

$$<\varphi\,;\,\psi> = \int_R \frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_i}dV$$

$\mathfrak{D}$ denotes the Hilbert product space $D_{\partial R'} \times \mathbf{D}_{\partial R}$, where $\mathbf{D}_{\partial R}$ is the vectorial counterpart of $D_{\partial R}$. In the norm and the scalar product are:

$$\begin{cases} <(\varphi,\ \underline{\phi})\ ;\ (\omega,\ \underline{\Omega}) = \displaystyle\int_R \left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\omega}{\partial x_i} + \frac{\partial\phi_j}{\partial x_i}\frac{\partial\Omega_j}{\partial x_i}\right)dV \\ \|(\varphi,\ \underline{\phi})\|_{\mathfrak{D}} = (<(\varphi,\ \underline{\phi})\ ;\ (\varphi,\ \underline{\phi})>)^{1/2} \end{cases}$$

$\mathfrak{D}$ and $\mathcal{W}^1_\partial$ consist of identical elements. Since both the spaces $\mathfrak{D}$ and $\mathcal{W}^1_\partial$ are product spaces it is enough to prove that the component spaces have identical elements.

Proof: The spaces $D_{\partial R'}$ and $W^\ell_{2,\partial R'}$ consist of identical elements since $\mathcal{C}^\infty$ is dense in both $D_{\partial R'}$ and $W^\ell_{2,\partial R'}$, and, in addition, the norms are equivalent.

$$\begin{cases} \|\varphi\|^2_{W^\ell_{2,\partial R'}} \equiv \displaystyle\int_R \left(\varphi^2 + \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\right)dV \underset{(F)}{\leq} (C^2+1)\int_R \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dV \equiv C^2+1)\|\varphi\|^2_{D_{\partial R'}} \\ \|\varphi\|^2_{D_{\partial R'}} \equiv \displaystyle\int_R \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dV \leqslant \int_R \left(\varphi^2 + \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\right)dV \equiv \|\varphi\|^2_{W^\ell_{2,\partial R'}} \end{cases}$$

or

$$\epsilon_1\ \|\varphi\|^2_{W^\ell_{2,\partial R'}} \leqslant \|\varphi\|^2_{D_{\partial R'}} \leqslant \epsilon\ \ \|\varphi\|^2_{W^\ell_{2,\partial R'}}$$

The proof for the vectorial part goes along the same lines.

The set **S** consist of all smooth, solenoidal vectors which vanish on ∂R. The completion of **S** in the L_2 norm is denoted by $L_{2,s}$.

In L_2, two vectors are said to be orthogonal if their scalar product vanishes. The orthogonal complement of $L_{2,s}$ (i.e, the set of all vectors $\underline{\phi}_p$ which belong to L_2 and are orthogonal to every $\underline{\phi}_s \in L_{2,s}$) consists of potential vectors [11 : §62] and is denoted by $L_{2,p}$.

Let **H** denote the completion of **S** in the **D** norm. The product space $D_{\partial R'} \times \mathbf{H}$ is denoted by Z. By definition, the norm and the scalar product in Z are:

$$\begin{cases} <(\varphi, \underline{\phi}) ; (\omega, \underline{\Omega})> = \int_R \left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\omega}{\partial x_i} + \frac{\partial\phi_i}{\partial x_j}\frac{\partial\Omega_i}{\partial x_j}\right) dV \\ \|(\varphi, \underline{\phi})\|_Z = (<(\varphi, \underline{\phi}) ; (\varphi, \underline{\phi})>)^{1/2} \end{cases}$$

In this space an obviously equivalent norm can be introduced:

$$\|(\varphi, \phi)\|_{Z_\epsilon} = (\epsilon^{-1/2} \|\varphi\|^2_{D_{\partial R'}} + \epsilon^{+1/2}\|\underline{\phi}\|^2_H)^{1/2}$$

The space Z normed in this way is denoted by Z_ϵ. For ϵ = Pr the space Z_ϵ is denoted Z_{Pr}. The stability problem is investigated in this space. The first element φ in $(\varphi, \underline{\phi}) \in Z_{Pr}$ may be thought of as a temperature field which "vanishes" on $\partial\bar{R}'$ and has "vanishing normal derivative" on $\partial R''$. The second element in $(\varphi, \underline{\phi})$ represents a velocity field which "vanishes" on ∂R.

6.3. Embedding Theorems: The following embedding theorems are either parts or direct correlaries of Sobolov Embedding Theorems [9 : § 114].

$$\begin{cases} W_2^\ell \subset W_2^{\ell-1} \subset \dots\dots \subset W_2^1 \subset W_2^o \equiv L_2 \\ \mathbf{W}_2^\ell \subset \mathbf{W}_2^{\ell-1} \subset \dots\dots \subset \mathbf{W}_2^1 \subset \mathbf{W}_2^o \equiv \mathbf{L}_2 \\ \mathcal{W}^\ell \subset \mathcal{W}^{\ell-1} \subset \dots\dots \subset \mathcal{W}^1 \subset \mathcal{W}^o \equiv \mathcal{L} \end{cases}$$

The space $\mathfrak{D}$ is embedded in $\mathcal{L}$:
All the elements of $\mathfrak{D}$ are identical with those of $\mathcal{W}^1_\partial$. $\mathcal{W}^1_\partial$ is a proper subspace of $\mathcal{W}^1$, which is embedded in $\mathcal{L}$

$$\mathfrak{D} \equiv \mathcal{W}^1_2 \subset \mathcal{W}^1 \subset \mathcal{L} .$$

The space **H** is embedded in $\mathbf{D}_{\partial R'}$:
The norm in both **H** and $\mathbf{D}_{\partial R}$ is the norm of **D**; the embedding follows from the fact that **S** is a proper subset of $\mathbf{D}_{\partial R}$ (vectors in both **S** and $\mathbf{D}_{\partial R}$ have components in $C'_{\partial R}$ but only solenoidal vectors are in **S**).

The space Z is embedded in $\mathfrak{D}$ since they both have $D_{\partial R'}$ as first component; the second component of Z, is embedded in $\mathbf{D}_{\partial R'}$, the second component of $\mathfrak{D}$.

The embedding of Z in $\mathcal{L}$ follows from the embedding of Z in $\mathfrak{D}$ and the embedding of $\mathfrak{D}$ in $\mathcal{L}$.

The space **H** is embedded in $\mathbf{L}_{2,S}$:
The set **S** is dense in both **H** and $\mathbf{L}_{2,S}$, and

$$\|\underline{\phi}\|_H^2 = \int_R \frac{\partial\phi_i}{\partial x_j}\frac{\partial\phi_i}{\partial x_j} dV \leqslant C^2 \int_R \phi_i\phi_i dV = C^2 \|\underline{\phi}\|_{L_2}^2 ;$$ i.e. the convergence in **H** follows from the convergence in L_2.

6.4. Operators in Hilberts spaces: Let E_1 and E_2 be Hilbert spaces and let S be an arbitrary set of E_1. The set of ordered pairs $\{(x, Ax)\}$, $x \in S$ and $A\,x \in E_2$, defines an operator A *from* E_1 *to* E_2 if there exists no pair in the set having identical first element and different second element. The "domain" of A is just the set S and the "range" of A is the set of all elements in E_2 of the form $A\,x$, $x \in S$. If E_1 is the domain of A the operator A is said to be defined *on* E_1; if, in addition, the spaces E_1 and E_2 are identical, A is said to be an operator *in* E_1.

The operator A is said to be *bounded* if the image of any bounded set in E_1 is a bounded set in E_2; i.e, $x_n \in S$ and $\|x_n\| < M$ imply $\|Ax_n\| < N$ where both M and N do not depend on n.

The operator A is said to be *continuous at* $x_0 \in S$ if the image of any sequence $\{Ax_n\}$ which converges to Ax_0; i.e, if $\lim_{n\to\infty} \|x_n - x_0\|_{E_1} = 0$ then $\lim_{n\to\infty} \|Ax_n - Ax_0\|_{E_2} = 0$. This kind of convergence is sometimes called strong convergence and denoted by $\Longrightarrow$. Since not other kind of convergence is used in the present work, the term strong is omitted.

Operators which are continuous on every point of E_1, are simply called continuous on E_1.

A bounded set $S \subset E_1$, is called *compact* in E_1 if any sequence of elements $\{x_n \in S\}$ contains a subsequence which converges in the norm of E_1. The operator A is called *compact* on a set $S \subset E_1$, if it takes every bounded subset of S into a compact set in the space E_2. An operator which is continuous and compact on $S \subset E_1$, is called *completely continuous* on S.

The operator I is called the identity operator in E if the image of every element $x \in E$ is the element x itself. Moreover, if $E_1 \subset E_2$ then the identity operator I on E_1 to E_2 is defined and it takes every element $x \in E_1$ to the same element x which is now regarded as an element of E_2.

The identity operator I on W_2^1 to L_2 is completely continuous [9 : §114]. Both the identity operator on $\mathbf{W}_2^1$ to $\mathbf{L}_2$ and the identity operator on $\mathcal{W}^1$ to $\mathcal{L}$ are completely continuous, because these spaces are products of a finite number of W_2^1 or respectively, L_2 spaces.

The operator A is called *distributive* on E if

$$A(\lambda x + \mu y) = \lambda A x + \mu A y,$$

for all $x, y \in E_1$, and all real numbers λ and μ. An operator which is both distributive and continuous is called *linear*.

Theorem: The distributive operator A is linear if and only if there exists a constant C such that $\|Ax\| \leqslant C^2 \|x\|$ for all $x \in E_1$ [9 : §97]. The inequality

$\|Ax\| \leqslant C^2 \|x\|$ guarantees A to be bounded.

Let A be a linear operator; if there is bounded operator B such that $AB = BA = I$, then B is called "*the inverse of A*" and is denoted by A^{-1}. The inverse operator is linear [9 : §127].

The set of all linear operators on E_1 to E_2 is a Banach space (satisfies a) to e) in the definition of Hilbert spaces) usually denoted $E_{1,2}$ [9 : § 104]. The norm of a linear operator, which is an element of $E_{1,2}$, satisfies:

$$\|A\| = \sup_{x \in E_1} \frac{\|A\, x\|}{\|x\|} = \sup_{x \in E_1, \|x\| \leqslant 1} \|A\, x\| = \sup_{x \in E_1, \|x\|=1} \|A\, x\|$$

Theorem [*9* ; §*136*]: Let A be a linear, completely continuous operator in E; then:
a) To every given ϵ, $\epsilon > 0$, there exists only a finite number of values λ, $\|\lambda\| < \epsilon$, such that the equation $Ax + \lambda x = 0$ has a non-zero solution. These solutions are called eigenvectors of A and the corresponding λ are called eigenvalues.
Corrolary of a): The set of all eigenvalues is at most countable.
b) The operator $(A - \lambda I)^{-1}$ exist for all regular values of λ(all values of λ which are not eigenvalues).
c) If, in additions, the operator A is symmetric; i.e.

$$< y \; ; \; Ax > = <Ay \; ; \; x> \text{ for all } x,\ y \in A,$$

then there exists at least one eigenvalue. Moreover, the highest eigenvalue satisfies:

$$|\lambda^+| = \sup_{x \in E_1 ; \|x\| = 1} <Ax \; ; \; x> = \|A\|$$

A linear operator from E_1 to E_2 is called a *linear functional* on E_1 if E_2 consists of all real numbers.

Riesz' Theorem [*11:* § *3*]: Every linear functional l on E can be written as a scalar product of a constant element $x_\ell \in E$ and the element $x \in E$; i.e.,

$$l(x) = <x_\ell \; ; \; x> \text{ for all } x \in E_1;$$

the element x_ℓ is unique.
An operator b acting on E_1 x E_2 to the real numbers system is called a bilinear functional if for every y, $y \in E_2$ the operator is a linear functional on E_1 and vice-versa, for all $x \in E_1$ the operator is a linear functional on E_2. Bilinear functionals are bounded operators because $|b(x,y)| \leqslant C^2 \|x\| \, \|y\|$, for all $(x,y) \in E_2$. The smallest value C^2 for which this inequality is still valid is called the norm of b, $\|b\|$.

$$\|b\| = \sup_{\|x\|=1, \|y\|=1} |b(x, y)|$$

From Riesz' Theorem follows [9 : §125] that each bilinear functional defines a unique linear operator A (or its conjugate A^*) given by:

$$b(x, y) , < Ax \; ; \; y> = <x \; ; \; A^* y>$$

(the last equality is the definition of the conjugate operator A^*). Moreover, the inverse is also true: every linear operator A defines a bilinear functional by means of the same expression.

A bilinear functional is called symmetric if

$$b(x, y) = b(y, x)$$

The operator defined by a symmetric bilinear functional is, of course, symmetric.

6.5. Frechet Derivative of Operators [*12 : 1 § 3.3*]*:* Let A be an operator on E_2 to E ; if, at the point $x_o \in E_1$

$$A(x_o + h) - A(x_o) = A_\ell h + A_r(h)$$

where A_ℓ is a linear operator on $h \in E_1$, and

$$\lim_{\|h\| \to 0} \frac{\|A_r(h)\|}{\|h\|} = 0$$

then $A_\ell h$ is called the Frechet differential of A at the point $x_o \in E_1$, and $A_r(h)$ is called the remainder of the differential. The Frechet derivative of the operator A is denoted by A'. It is the operator from E_1 to $E_{1,2}$ which takes elements $x \in E_1$, on which the Frechet differential is defined, to corresponding linear operator $A_\ell \in E_{1,2}$. For clarity two examples are given:

Example 1: Let A be the operator defined by

$$A = \{(f \in C,\ f^3)\}$$

By definition, for every $f_o \in C$

$$A_\ell = \{(h \in C,\ 3f_o^2 h)\} \ ; \ A_r = \{(h \in C,\ 3f_o h^2 + h^3)\}$$

i.e. the function $3f_o^2 h \in C$ is the Frechet differential on f_o in the direction of h, and the operator

$$A_o^1 = \{h \in C,\ 3f_o^2 h\}$$

is the derivative of A on f_o. The derivative of A is the operator A'

$$A^1 = \{(f \in C,\ (h \in C,\ 3f_o^2 h))\}$$

Example 2: Let A be defined on **H** by:

$$A = \{(\underline{V} \in \mathbf{H},\ (\underline{V} \cdot \nabla)\underline{V})\}$$

By definition, on $\underline{V}_o \in \mathbf{H}$

$$\begin{cases} A_\ell = \{(\underline{h} \in \mathbf{H},\ (\underline{V}_o \cdot \nabla)\underline{h} + (\underline{h} \cdot \nabla)V_o)\} \\ A_r = \{(\underline{h} \in \mathbf{H},\ (\underline{h} \cdot \nabla)\underline{h})\} \end{cases}$$

The Frechet derivative of A is the operator A'.

$$A' = \{(\underline{V} \in \mathbf{H},\ (\underline{h} \in \mathbf{H},\ (\underline{V} \cdot \nabla)\underline{h} + (\underline{h} \cdot \nabla)\underline{V}))\}$$

The Theorem of Hildebrand and Graves: Let B be an operator taking pairs (x, y), $x \in E_1$, $y \in E_2$ into a space E_3. Further, suppose that $B(x_0, y_0) = Q$ for some (x_0, y_0), that B is continuous with respect to (x, y) in some neighborhood of (x_0, y_0), and has in that neighborhood a (partial) Frechet derivative with respect to x which is continuous in that neighborhood with respect to (x, y); let B at the point (x_0, y_0), have a linear inverse operator. Then, in the neighborhood of (x_0, y_0), the equation $B(x, y) = Q$ has a unique solution for every y [12 : 5 §17].

7. *Intermediate Results*

Theorem: Let $\underline{\Omega} \in H$ be given, and let $(\varphi, \underline{\phi})$, $(\psi, \underline{\psi}) \in Z$. The integral $b_1 = \int_R \varphi\, \underline{\Omega} \cdot \underline{\psi}\, dV$ defines a bilinear functional on $Z \times Z$.

Proof: The distributive properties with respect to each of the elements $(\varphi, \underline{\phi})$ or $(\psi, \underline{\psi})$ is obvious. To prove the boundedness consider*)

$$\left| \int_R \varphi\, \underline{\Omega} \cdot \underline{\psi}\, dV \right| = \left| \int_R \varphi\, \Omega_i \psi_i\, dV \right| \underset{(H)}{\leq} \left(\int \varphi^2\right)^{1/2} \left(\int \Omega_i^4\right)^{1/4} \left(\int \psi_i^4\right)^{1/4} =$$

$$\underset{(L)}{\leqslant} 2\left(\int \varphi^2\right)^{1/2} \left(\int \Omega_i^2\right)^{1/8} \left(\int \frac{\partial \Omega_i}{\partial x_j} \frac{\partial \Omega_i}{\partial x_j}\right)^{3/8} \left(\int \psi_i^2\right)^{1/8} \left(\int \frac{\partial \psi_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j}\right)^{3/8}$$

By addition of positive term to the right hand term, this becomes:**)

$$\left| \int_R \varphi\, \underline{\Omega} \cdot \underline{\psi}\, dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_{\mathcal{L}}\ \|\underline{\Omega}\|_L^{1/4} \|\underline{\Omega}\|_H^{3/4} \|(\psi, \underline{\psi})\|_{\mathcal{L}}^{1/4} \|(\psi, \underline{\psi})\|_Z^{3/4} \tag{11a}$$

Because $Z \subset \mathcal{L}$ ($\| \ \|_{\mathcal{L}} \leqslant \epsilon \| \ \|_Z$), and Ω is a fixed point, this yields

$$\left| \int_R \varphi\, \underline{\Omega} \cdot \underline{\psi}\, dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_{\mathcal{L}} \|(\psi, \underline{\psi})\|_Z < C^2 \|(\varphi, \underline{\phi})\|_Z \|(\psi, \underline{\psi})\|_Z \tag{12}$$

It is important to note that everywhere in these inequalities the elements $(\varphi, \underline{\phi})$ and $(\psi, \underline{\psi})$ are interchangable:

$$\left| \int_R \varphi\, \underline{\Omega} \cdot \underline{\psi}\, dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_{\mathcal{L}}^{1/4} \|(\varphi, \underline{\phi})\|_Z^{3/4} \|\underline{\Omega}\|_L^{1/4} \|\underline{\Omega}\|_H^{3/4} \|(\psi, \underline{\psi})\|_{\mathcal{L}} \tag{11b}$$

Theorem: Let $\underline{\Omega} \in H$ be given and let $(\varphi, \underline{\phi})$, $(\psi, \underline{\psi}) \in Z$. The integral

*) In proofs, the domain of the integration, always R, and the volume element will be omitted. $\underset{(H)}{\leqslant}$ means, from (H) inequality etc.

**) All constants are denoted by the same C as long as their numerical values are irrelevant.

$b_2 = \int_R \varphi \underline{\Omega} \cdot \nabla \psi dV$ is a bilinear functional on $Z \times Z$.

Proof: The distributiveness with respect to each element $(\varphi, \underline{\phi})$ or $(\psi, \underline{\psi})$ is obvious. To prove the boundedness consider:

$$\left|\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV\right| \equiv \left|\int_R \varphi \Omega_i \frac{\partial \psi}{\partial x_i} dV\right| \underset{(H)}{\leq} \left(\int \varphi^4\right)^{1/4} \left(\int \Omega_i^4\right)^{1/4} \left[\int \left(\frac{\partial \psi}{\partial x_i}\right)^2\right]^{1/2}$$

$$\underset{(L)}{\leqslant} 2 \left(\int \varphi^2\right)^{1/8} \left(\int \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_j}\right)^{3/8} \left(\int \Omega_i^2\right)^{1/8} \left(\int \frac{\partial \Omega_j}{\partial x_k} \frac{\partial \Omega_j}{\partial x_k}\right)^{3/8} \left[\int \left(\frac{\partial \psi}{\partial x_i}\right)^2\right]^{1/2}$$

The addition of positive terms to the right hand term gives:

$$\left|\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV\right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_{\mathcal{L}}^{1/4} \|(\varphi, \underline{\phi})\|_Z^{3/4} \|\Omega\|_{L_2}^{1/4} \|\Omega\|_H^{3/4} \|(\psi, \underline{\psi})\|_Z \tag{13}$$

Because $Z \subset \mathcal{L}$ and $\underline{\Omega}$ is a fixed element then

$$\left|\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV\right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|(\psi, \underline{\psi})\|_Z \tag{14}$$

i.e, b_2 is bounded.

Now, $\underline{\Omega} \in \mathbf{H}$ means $\underline{\Omega}|_{\partial R} = 0$ and $\nabla \cdot \underline{\Omega} = 0$; therefore the bilinear functional is skew symmetric

$$\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV = -\int_R \psi \underline{\Omega} \cdot \nabla \varphi dV$$

i.e, $(\varphi, \underline{\phi})$ and $(\psi, \underline{\psi})$ are interchangeable.

$$\left|\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV\right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|\underline{\Omega}\|_{L_2}^{1/4} \|\underline{\Omega}\|_H^{3/4} \|(\psi, \underline{\psi})\|_{\mathcal{L}}^{1/4} \|(\psi, \underline{\psi})\|_Z^{3/4} \tag{15}$$

The element $\underline{\Omega} \in \mathbf{H}$ can be considered an element of Z, $(\omega, \underline{\Omega}) \in Z$, with arbitrary $\omega \in D_{\partial R'}$. With this interpretation inequalities 13) and 15) become:

$$\left|\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV\right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_{\mathcal{L}}^{1/4} \|(\varphi, \underline{\phi})\|_Z^{3/4} \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \|(\omega, \underline{\Omega})\|_Z^{3/4} \|(\psi, \underline{\psi})\|_Z \tag{16}$$

$$\left|\int_R \varphi \underline{\Omega} \cdot \nabla \psi dV\right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \|(\omega, \underline{\Omega})\|_Z^{3/4} \|(\psi, \underline{\psi})\|_{\mathcal{L}}^{1/4} \|(\psi, \underline{\psi})\|_Z^{3/4} \tag{17}$$

Corrolary: The integral $b_3 = \int_R \underline{\phi} \cdot \left[(\underline{\Omega} \cdot \nabla)\underline{\psi}\right] dV$ defines a bilinear functional

functional on $Z \times Z$, because b_3 is a sum of finite number of bilinear functionals of the form of b_2.

The following inequalities are direct consequences of the definition of and the properties of b_2, Eqs. 13) ÷ 17):

$$\left| \int_R \underline{\phi} \cdot [(\underline{\Omega} \cdot \nabla)\underline{\psi}] \, dV \right| \leqslant C^2 \, \|(\varphi, \underline{\phi})\|_{\mathcal{L}}^{1/4} \, \|(\varphi, \underline{\phi})\|_Z^{3/4} \, \|\underline{\Omega}\|_{L_2}^{1/4} \, \|\underline{\Omega}\|_H^{3/4} \, \|(\psi, \underline{\psi})\|_Z \tag{18}$$

$$\left| \int_R \underline{\phi} \cdot [(\underline{\Omega} \cdot \nabla)\underline{\psi}] \, dV \leqslant C^2 \, \|(\varphi, \underline{\phi})\|_Z \, \|(\psi, \underline{\psi})\|_Z \tag{19}$$

$$\left| \int_R \underline{\phi} \cdot [(\underline{\Omega} \cdot \nabla)\underline{\psi}] \, dV \leqslant C^2 \, \|(\varphi, \underline{\phi})\|_Z \, \|\underline{\Omega}\|_{L_2}^{1/4} \, \|\underline{\Omega}\|_H^{3/4} \, \|(\psi, \underline{\psi})\|_{\mathcal{L}}^{1/4} \, \|(\psi, \underline{\psi})\|_Z^{3/4} \tag{20}$$

$$\left| \int_R \underline{\phi} \cdot [(\underline{\Omega} \cdot \nabla)\underline{\psi}] \, dV \leqslant C^2 \, \|(\varphi, \underline{\phi})\|_{\mathcal{L}}^{1/4} \, \|(\varphi, \underline{\phi})\|_Z^{3/4} \, \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \, \|(\omega, \underline{\Omega})\|_Z^{3/4} \, \|(\psi, \underline{\psi})\|_Z \tag{21}$$

$$\left| \int_R \underline{\phi} [(\underline{\Omega} \cdot \nabla)\underline{\psi}] \, dV \leqslant C^2 \, \|(\varphi, \underline{\phi})\|_Z \, \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \, \|(\omega, \underline{\Omega})\|_Z^{3/4} \, \|(\psi, \underline{\psi})\|_{\mathcal{L}}^{1/4} \, \|(\psi, \underline{\psi})\|_Z^{3/4} \tag{22}$$

Theorem: Let K_ℓ be the linear operator associated with the linear functional

$$- \int_R \varphi G \cdot \underline{\phi} \, dV - \int_R \underline{\psi} \cdot \nabla \overset{o}{T}_\infty \varphi \, dV$$

i.e, for all $(\varphi, \underline{\phi})$, $(\psi, \underline{\psi}) \in Z$

$$<(\varphi, \underline{\phi}); K(\psi, \underline{\psi})> = - \int_R \varphi \underline{G} \cdot \underline{\phi} \, dV - \int_R \underline{\psi} \cdot \nabla \overset{o}{T}_\infty \varphi \, dV \tag{23}$$

then, K_ℓ is completely continuous.

Proof: The continuity of K_ℓ follows from its linearity. To prove the compactness let S, $S \subset Z$, be a bounded set, and let $\{(\omega_n, \underline{\Omega}_n)\}$ be a sequence of elements in S. Because $Z \subset \mathfrak{D} \subset \mathcal{W}_2^1$, this sequence can be considered a sequence in $\mathcal{W}_2^1$. The identity operator on $\mathcal{W}_2^1$ to $\mathcal{L}$ is compect and therefore the sequence $\{(\omega_n, \underline{\Omega}_n) \in \mathcal{W}_2^1\}$ containes a subsequence which converges in the norm $\mathcal{L}$ norm (see §6.4.).

For simplicity the subsequence is also denoted by $\{(\omega_n, \underline{\Omega}_n)\}$.

The convergence of $\{K_\ell(\omega_n, \underline{\Omega}_n)\}$ (the image of the convergent subsequence) follows from:

a) Since both $\underline{G}_\infty$ and $\nabla \overset{o}{T}_\infty$ are fixed elements (see Eq. 12))

$$\begin{cases} \left| \int_R \omega \underline{G} \cdot \underline{\phi} \, dV \right| \leqslant C^2 \, \|(\omega, \underline{\Omega})\| \, \|(\varphi, \underline{\phi})\|_Z \\ \left| \int_R \varphi \underline{\Omega} \cdot \nabla \mathring{T}_\infty \, dV \right| \leqslant C^2 \, \|(\omega, \underline{\Omega})\| \, \|(\varphi, \underline{\phi})\|_Z \end{cases}$$

b) By definition a) leads to

$$| <(\varphi, \underline{\phi}); K_\ell(\omega, \underline{\Omega})> | \leqslant C^2 \, \|(\omega, \underline{\Omega})\| \, \|(\varphi, \underline{\phi})\|_Z \quad \text{for all } (\varphi, \underline{\phi}) \in Z$$

or, by substituting $(\varphi, \underline{\phi}) = K_\ell(\omega, \underline{\Omega})$

$$\|K_\ell(\omega, \underline{\Omega})\|_Z \leqslant C^2 \, \|(\omega, \underline{\Omega})\|_{\mathcal{L}}$$

c) Because K_ℓ is linear, and because of a) and b) and the convergence of $\{(\omega_n, \Omega_n)\}$ in the $\mathcal{L}$ norm, then

$$\lim_{m,n\to\infty} \| K_\ell(\omega_m, \underline{\Omega}_m) - K_\ell(\omega_n, \underline{\Omega}_n)\|_Z = \lim_{m,n\to\infty} \| K_\ell(\omega_m, \underline{\Omega}_m) - (\omega_n, \underline{\Omega}_n)\|_Z =$$

$$\leqslant C^2 \lim_{m,n\to\infty} \|(\omega_m, \underline{\Omega}_m) - (\omega_n, \Omega_n)\|_{\mathcal{L}} = 0$$

This inequality is, by definition, the condition for convergence of $\{K_\ell(\omega_n, \underline{\Omega}_n)\}$ in Z.

Theorem: Let $(\omega, \underline{\Omega}) \in Z$ be a fixed element and let P_ℓ be the linear operator defined by:

$$<(\varphi, \underline{\phi}); P_\ell(\psi, \underline{\psi})> = - \int_R \varphi \underline{\Omega} \cdot \nabla \psi \, dV - \int_R \varphi \underline{\psi} \cdot \nabla \omega \, dV +$$

$$- \int_R \underline{\phi} \cdot \left[(\underline{\Omega} \cdot \nabla) \underline{\psi} \right] dV - \int_R \underline{\phi} \cdot \left[(\underline{\psi} \cdot \nabla) \underline{\Omega} \right] dV \tag{24}$$

for all $(\varphi, \underline{\phi}), (\psi, \underline{\psi}) \in Z$

then, P_ℓ is completely continuous.

Proof: The continuity of P_ℓ follows from the linearity. To proce the compactness let S, $S \subset Z$, be a bounded set and let $\{(\psi_n, \underline{\psi}_n)\}$ be a sequence in S. As in the previous theorem this sequence can be a-priory chosen to converges in $\mathcal{L}$. Similarly, the convergence of $\{P_\ell(\psi_n, \underline{\psi}_n)\}$ follows from:
a) The first two integrals in Eq. (24) are bilinear functionals of the from of b_2; hence, (see Eq. 17)

$$\begin{cases} \left| \int_R \varphi \, \underline{\Omega} \cdot \nabla \psi_n \, dV \right| \; C^2 \, \|(\varphi, \underline{\phi})\|_Z \, \|(\omega_n, \underline{\psi}_n)\|_{\mathcal{L}}^{1/4} \, \|(\psi_n, \underline{\psi}_n)\|_Z^{3/4} \, \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \, \|(\omega, \underline{\Omega})\|_Z^{3/4} \\ \left| \int_R \varphi \, \underline{\psi}_n \cdot \nabla \omega \, dV \right| \leqslant C^2 \, \|(\varphi, \underline{\phi})\|_Z \, \|(\omega_n, \underline{\psi}_n)\|_{\mathcal{L}}^{1/4} \, \|(\omega_n, \underline{\psi}_n)\|_Z^{3/4} \, \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} (\omega, \underline{\Omega})\|_Z^{3/4} \end{cases} \tag{25a}$$

b) The other integrals in Eq. (24) are bilinear functionals of the form of b_3; hence, (see Eq. 22)

$$\begin{cases} \left| \int_R \underline{\phi} \cdot [(\underline{\Omega}\cdot\nabla)\underline{\psi}_n]\, dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|(\psi_n, \underline{\psi}_n)\|_{\mathcal{L}}^{1/4} \|(\psi_n, \underline{\psi}_n)\|_Z^{3/4} \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \|(\omega, \underline{\Omega})\|_Z^{3/4} \\ \left| \int_R \underline{\phi} \cdot [(\underline{\psi}_n\cdot\nabla)\underline{\Omega}]\, dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|(\psi_n, \underline{\psi}_n)\|_{\mathcal{L}}^{1/4} \|(\psi_n, \underline{\psi}_n)\|_Z^{3/4} \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \|(\omega, \underline{\Omega})\|_Z^{3/4} \end{cases} \qquad (25b)$$

c) Substitution of $(\varphi, \underline{\phi}) = P_{\ell}(\psi_n, \underline{\psi}_n)$ in (25a) and (25b) above, and because $(\omega, \underline{\Omega})$ is a fixed element:

$$\|P_{\ell}(\psi_n, \underline{\psi}_n)\|_Z \leqslant C^2 \|(\psi_n, \underline{\psi}_n)\|_{\mathcal{L}}^{1/4} \|(\psi_n, \underline{\psi}_n)\|_Z^{3/4} \qquad (26)$$

d) Because S is bounded $\|(\psi_n, \underline{\psi}_n)\|_Z^{3/4} < C^2$

$$\|P_{\ell}(\psi_n, \underline{\psi}_n)\|_Z \leqslant C^2 \|(\psi_n, \underline{\psi}_n)\|_{\mathcal{L}}^{1/4}$$

e) Because P_{ℓ} is linear d) becomes

$$\lim_{m,n\to\infty} \|P_{\ell}(\psi_n, \underline{\psi}_n) - P_{\ell}(\psi_m, \underline{\psi}_m)\|_Z = \lim_{m,n\to\infty} \|P_{\ell}\{(\psi_m, \underline{\psi}_m) - (\psi_n, \underline{\psi}_n)\}\|_Z$$

$$\leqslant C^2 \lim_{m,n\to\infty} \|(\psi_m, \underline{\psi}_m) - (\psi_n, \underline{\psi}_n)\|_{\mathcal{L}} = 0,$$

which means that $\{P_{\ell}(\psi_n, \underline{\psi}_n)\}$ is convergent in Z.

The operator P_{ℓ} can be considered as determined by the element $(\omega, \underline{\Omega})$. Because of the symmetry in the positions of $(\omega, \underline{\Omega})$ and $(\omega, \underline{\psi})$, P_{ℓ} can be, alternatively, considered to be determined by $(\psi, \underline{\psi})$; therefore $(\omega, \underline{\Omega})$ and $(\psi, \underline{\psi})$ are interchangeable in the previous inequalities.

Theorem: Let the operator K_s in Z*) be defined by the following relation:

$$<(\varphi, \underline{\phi})\ ;\ K_s(\omega, \underline{\Omega})> = - \int_R \underline{\phi}\cdot[(\underline{\Omega}\cdot\nabla)\underline{\Omega}]dV - \int \varphi\, \underline{\Omega}\cdot\nabla\, \omega dV$$

for all $(\omega, \underline{\Omega}) \in Z$

Then: 1) K_s is properly defined as an operator,
2) K_s is bounded,
3) K_s is continuous,
4) K_s is compact,

i.e, K_s is a completely continuous operator in Z.

Proof: 1) Let $(\omega, \underline{\Omega}) \in Z$ be a given fixed element. The righthand side integrals are, obviously, a linear functional $l(\varphi, \underline{\phi})$. This linear functional is formed by the addition of a bilinear functional of the form of b_2 to a bilinear functional of the form of b_3, in both of which the second element has been held constant. This constant element was made to equal the defining element. Therefore (see Eq. 17 and Eq. 22)

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*) Because Z and Z_{Pr} have only identical elements and their norms are equivalent all operators can be, alternatively, considered as defined on Z_{Pr}.

$$\begin{cases} \left| \int_R \varphi \underline{\Omega} \cdot \nabla \omega \, dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/2} \|(\omega, \underline{\Omega})\|_Z^{3/2} \\ \left| \int_R \underline{\phi} \cdot \left[(\underline{\Omega} \cdot \nabla) \underline{\Omega} \right] dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \|(\omega, \underline{\Omega})\|_{\mathcal{L}}^{1/2} \|(\omega, \underline{\Omega})\|_Z^{3/2} \end{cases} \tag{27}$$

or, because $(\omega, \underline{\Omega})$ is fixed in Z:

$$\left| \int_R \varphi \underline{\Omega} \cdot \nabla \omega dV + \int_R \underline{\phi} \cdot \left[(\underline{\Omega} \cdot \nabla) \Omega \right] dV \right| \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z \tag{28}$$

Now, from Riesz' theorem, the element $K_s(\omega, \underline{\Omega})$ is uniquely defined by the linear functional $l(\varphi, \underline{\phi})$; but $l(\varphi, \underline{\phi})$ is determined by the fixed element $(\omega, \underline{\Omega})$; and, therefore, the set of pairs $\{(\omega, \underline{\Omega}); K_s(\omega, \underline{\Omega})\}$ defines the operator K_s.

2) The boundedness of K_s is, obviously a consequence of (28).

3) Let (ω_n, Ω_n) be a sequence which converges to $(\omega_o, \Omega_o) \in Z$. By definition:

$$<(\varphi, \underline{\phi}); K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)> = - \int_R \varphi(\underline{\Omega}_n \cdot \nabla \omega_n - \underline{\Omega}_m \cdot \nabla \omega_m) dV +$$

$$- \int_R \underline{\phi} \cdot \left[(\underline{\Omega}_n \cdot \nabla) \underline{\Omega}_n - (\underline{\Omega}_m \cdot \nabla) \underline{\Omega}_m \right] dV$$

or, by addition and substraction of identical terms (see Eqs. 17 & 22)

$$<(\varphi, \underline{\phi}); K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)> \leqslant C^2 \|(\varphi, \underline{\phi})\|_Z (\|(\omega_n, \underline{\Omega}_n)\|_{\mathcal{L}}^{1/4} \|(\omega_n, \underline{\Omega}_n)\|_Z^{3/4} +$$
$$+ \|(\omega_m, \underline{\Omega}_m)\|_{\mathcal{L}}^{1/4} \|(\omega_m, \underline{\Omega}_m)\|_Z^{3/4}) \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)\|_{\mathcal{L}}^{1/4} \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)\|_Z^{3/4}$$

Substition of $(\varphi, \underline{\phi}) = K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)$ in the last inequality leads to

$$\|K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)\| \leqslant C^2 (\|(\omega_n, \underline{\Omega}_n)\|_{\mathcal{L}}^{1/4} \|(\omega_m, \underline{\Omega}_m)\|_Z^{3/4} +$$
$$+ \|(\omega_m, \underline{\Omega}_m)\|_{\mathcal{L}}^{1/4} \|(\omega_m, \underline{\Omega}_m)\|_Z^{3/4}) \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)\|_{\mathcal{L}}^{1/4} \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)\|_Z^{3/4} \tag{29}$$

and because of $Z \subset \mathcal{L}$ this becomes

$$\|K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)\|_Z \leqslant C^2 (\|(\omega_n, \underline{\Omega}_n)\|_Z + \|(\omega_m, \underline{\Omega}_m)\|_Z) \|(\omega_n, \underline{\Omega}_n) -$$
$$- (\omega_m, \underline{\Omega}_m)\|_Z$$ and $\{K_s(\omega_n, \underline{\Omega}_n) \in Z\}$ converges.

4) let $(\omega_n, \underline{\Omega}_n)$ be a sequence in S ($S \in Z$ and S bounded). It has already been shown that such a sequence can be chosen to converge in $\mathcal{L}$.

There exists an M, independent of n, such that $\|(\omega_n, \Omega_n)\|_Z \leqslant M$ and therefore, from Eq. 29):

$$\|K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)\|_Z \leqslant C^2 M^{7/4} \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)\|_{\mathcal{L}}^{1/4}$$

Because the sequence $(\omega_n, \underline{\Omega}_n)$ converges in $\mathcal{L}$

$$\lim_{m,n\to\infty} \| K_s(\omega_n, \underline{\Omega}_n) - K_s(\omega_m, \underline{\Omega}_m)\|_Z = 0$$

hence the sequence $\{K_s(\omega_n, \underline{\Omega}_n)\}$ converges in Z.

Theorem: The operator $K = K_\ell + K_s$ has a continuous Frechet derivative in some neigborhood of Q.

Proof: Let $(\omega, \underline{\Omega}) \in Z$ be a fixed element in some neighborhood of Q and let $(h, \underline{H}) \in Z$; by definition:

$$\begin{cases} \langle(\varphi, \underline{\phi}); K(\omega, \underline{\Omega})\rangle = -\int_R \omega \underline{G}_\infty \cdot \underline{\phi}\, dV - \int_R \varphi \underline{\Omega} \cdot \nabla \mathring{T}_\infty + \\ - \int_R \underline{\phi} \cdot \left[(\underline{\Omega} \cdot \nabla)\underline{\Omega}\right] dV - \int_R \varphi \underline{\Omega} \cdot \nabla \omega\, dV \\ \langle(\varphi, \underline{\phi}); K((\omega, \underline{\Omega}) + (h, \underline{H}))\rangle = -\int_R (\omega + h)\underline{G}_\infty \cdot \underline{\phi}\, dV - \int_R \varphi(\underline{\Omega} + \underline{H}) \cdot \nabla \mathring{T}_\infty\, dV + \\ - \int_R \underline{\phi} \cdot \left\{\left[(\underline{\Omega} + \underline{H}) \cdot \nabla\right](\underline{\Omega} + \underline{H})\right\} dV - \int_R \varphi(\underline{\Omega} + \underline{H}) \cdot \nabla(\omega + h) dV \end{cases}$$

and, therefore

$$\langle(\varphi, \underline{\phi}), K((\omega, \underline{\Omega}) + (h, \underline{H})) - K(\omega, \underline{\Omega})\rangle = -\int_R h \underline{\phi} \cdot \underline{G}_\infty dV - \int_R \varphi \underline{H} \cdot \nabla \mathring{T}_\infty dV +$$

$$- \int_R \underline{\phi} \cdot \left[(\underline{H} \cdot \nabla)\underline{H}\right] dV - \int_R \varphi \underline{H} \cdot \nabla h\, dV - \int_R \underline{\phi} \cdot \left[(\underline{H} \cdot \nabla)\underline{\Omega}\right] dV +$$

$$- \int_R \underline{\phi} \cdot \left[(\underline{\Omega} \cdot \nabla)\underline{H}\right] dV - \int_R \varphi \underline{\Omega} \cdot \nabla h\, dV - \int_R \varphi \underline{H} \cdot \nabla \omega\, dV$$

for all $(\varphi, \underline{\phi}) \in Z$

or,

$$\langle(\varphi, \underline{\phi}); K((\omega, \underline{\Omega}) + (h, \underline{H})) - K(\omega, \underline{\Omega})\rangle = \langle(\varphi, \underline{\phi}); K_\ell(h, \underline{H})\rangle +$$
$$+ \langle(\varphi, \underline{\phi}); P_\ell(h, \underline{H})\rangle + \langle(\varphi, \underline{\phi}), K_s(h, \underline{H})\rangle$$

for all $(\varphi, \underline{\phi}) \in Z$

Because $(\varphi, \underline{\phi}) \in Z$ is arbitrary and because Z is complete

$$K\{(\omega, \underline{\Omega}) + (h, \underline{H})\} - K(\omega, \underline{\Omega}) = (K_\ell + P_\ell)(h, \underline{H}) + K_s(h, \underline{H}).$$

$(K_\ell + P_\ell)(h, \underline{H})$ is identified now as the differential of K on the element $(\omega, \underline{\Omega})$ in the direction of $(h, \underline{H})$, and $K_\ell(L, \underline{H})$ as the remainder.

The operator K^1 which takes elements $(\omega, \underline{\Omega})$ from some neighborhood of Q to the operator $K_\ell + P_\ell$ (P_ℓ defined on the given $(\omega, \underline{\Omega})$ is the Frechet derivative of K. On the element Q, the derivative K_0^1 has the value of K_ℓ because P_ℓ defined on Q is, obviously, the null operator.

To prove the continuity of K^1 let $\{(\omega_n, \underline{\Omega}_n)\}$ be a sequence which converges to $(\omega_0, \underline{\Omega}_0) \in Z$ and let K_n^1 denote the value of the derivative of K on the element $(\omega_n, \underline{\Omega}_n)$. Because K_ℓ is independent of $(\omega_n, \underline{\Omega}_n)$

$$\|K'_m - K'_n\| = \|K_\ell + P_{\ell m} - (K_\ell + P_{\ell n})\| = \|P_{\ell m} - P_{\ell n}\| = \|P_{\ell(m-n)}\| \qquad (30)$$

where $P_{\ell m}$ is the operator P_ℓ defined on $(\omega_n, \underline{\Omega}_n)$ and $P_{\ell(m-n)}$ is, of course P_ℓ defined on $(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)$. From the previous inequalities (see Eq. 25)) follows that there exists a constant C^2 such that:

$$\|P_{\ell(m-n)}\| = \sup_{\|(h,\underline{H})\|=1} \|P_{\ell(m-n)}(h, \underline{H})\| \leqslant C^2 \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega})\|_{\mathcal{L}}^{1/4} \|(\omega_n, \underline{\Omega}_n) -$$
$$-(\omega_m, \underline{\Omega}_m)\|_Z^{3/4}$$

and because $Z \subset \mathcal{L}$

$$\|P_{\ell(m-n)}\| \leqslant C^2 \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \Omega_m)\|_Z$$

The convergence of $\{(K_n^1)\}$ follows from the convergence of $(\omega_n, \underline{\Omega}_n)$ in Z and from Eq. 30); i.e.,

$$\lim_{m,n\to\infty} \|K'_m - K'_n\| = \lim_{n,m\to\infty} \|P_{\ell(m-n)}\| \leqslant C^2 \lim_{m,n\to\infty} \|(\omega_n, \underline{\Omega}_n) - (\omega_m, \underline{\Omega}_m)\|_Z = 0$$

Theorem: Let $(\varphi, \underline{\phi}) \in Z$ and let λ be a regular point of K_ℓ (not an eigenvalue, see § 6.4). The solution of the equation
$0 = B(\varphi, \underline{\phi}) = K(\varphi, \underline{\phi}) - \lambda I(\varphi, \underline{\phi})$ is unique in some neighborhood of Q. This solution is

$$(\varphi, \underline{\phi}) = Q$$

Proof: From the properties of the operator K follows that in some neighborhood of Q the operator B has a continuous Frechet derivative B' (obviously, I' exist). Moreover, $B'_Q \equiv K'_Q - \lambda I'_Q = K_\ell - \lambda I$. Because λ is a regular point of K_ℓ, the inverse operator $(K_\ell - \lambda I)^{-1}$ exists and is linear (see § 6.4). Hence, from the Hildebrand - Graves' theorem follows that the solution of the equation $B(\varphi, \underline{\phi}) = Q$ is unique in some neigborhood of Q. From the definition of K follows $KQ = Q$; hence

$$B(Q) \equiv K(Q) - \lambda IQ = Q$$

8. *Results*

Theorem I: The solutions of Eq. (5) approach zero asymptotically as

$t \to \infty$ if $\|K_\ell\|_{Z_{pr}} < 1$.

Proof: Because $\underline{G}$ and $\nabla\overset{o}{T}$ approach $\underline{G}_\infty$ and $\nabla\overset{o}{T}_\infty$ asymptotically, there exist two functions $f_1(t)$ and $f_2(t)$ such that:

$$\sum_1^3 \int_R (\overset{o}{T} G_i - \overset{o}{T}_\infty G_{i\infty})^2 \, dV = f_1(t)$$

$$\sup_{\substack{\text{in } R\\ \text{on } i}} \left|(G_i - G_{i\infty}) + \frac{\partial}{\partial x_i}(\overset{o}{T} - \overset{o}{T}_\infty)\right| = f_2(t)$$

The functions $f_1(t)$ and $f_2(t)$ approach zero as $t \to \infty$. Let $(\theta, \underline{q})$, $(\theta, \underline{q}) \in Z_{Pr}$, be a solution of Eq. (5).
Then:

1.

$$\left|\int_R (\overset{o}{T} G_i - \overset{o}{T}_\infty G_i) q_i \, dV\right| \underset{(CBS)}{\leqslant} \left[\int_R (\overset{o}{T} G_i - \overset{o}{T}_\infty G_{i\infty})^2 dV\right]^{1/2} \left(\int_R q_i^2 \, dV\right)^{1/2}$$

$$\left|\int_R \theta \left[(G_i - G_{i\infty}) \frac{\partial}{\partial x_i}(\overset{o}{T} - \overset{o}{T}_\infty)\right] q_i \, dV\right| \underset{(CBS)}{\leqslant} \left\{\int_R \theta^2 \left[(G_i - G_{i\infty}) + \frac{\partial}{\partial x_i}(\overset{o}{T} - \overset{o}{T}_\infty)\right]^2 dV\right\}^{1/6} \cdot$$

$$\cdot \left(\int_R q_i^2 \, dV\right)^2 \leqslant \sup_{\substack{\text{in } R\\ \text{on } i}} \left|(G_i - G_{i\infty}) + \frac{\partial}{\partial x_i}(\overset{o}{T} - \overset{o}{T}_\infty)\right| \left(\int_R \theta^2 dV\right)^{1/2} \cdot \left(\int_R q_i q_i \, dV\right)^{1/2}$$

By the addition of positive terms on the righthand side these become:

$$\left|\int_R (\overset{o}{T} G_i - \overset{o}{T}_\infty G_{i\infty}) q_i \, dV\right| \leqslant f_1(t) \, \|(\theta, \underline{q})\|_{\mathcal{L}}$$

$$\left|\int_R \theta \left[(G_i - G_{i\infty}) + \frac{\partial}{\partial x_i}(\overset{o}{T} - \overset{o}{T}_\infty)\right] q_i \, dV\right| \leqslant f_2^2(t) \, \|(\theta, \underline{q})\|_{\mathcal{L}}^2 \tag{31}$$

2. By definition

$$\left|\int_R \theta(\underline{G}_\infty + \nabla \overset{o}{T}_\infty) \cdot \underline{q} \, dV\right| = \left|<(\theta, \underline{q}) \, ; \, K_\ell(\theta, \underline{q})>\right| =$$

$$\leqslant \|K_\ell(\theta, \underline{q})\|_{Z_{Pr}} \, \|(\theta, \underline{q})\|_{Z_{Pr}}$$

and because of the linearity of K_ℓ

$$\left|\int_R \theta(\underline{G}_\infty + \nabla \overset{o}{T}_\infty) \cdot \underline{q} \, dV\right| \leqslant \|K_\ell\|_{Z_{Pr}} \, \|(\theta, \underline{q})\|_{Z_{Pr}}^2$$

3. From the above inequality and from Eqs. (18) and (31)

$$\frac{1}{2}\frac{d}{dt}\|(\theta, \underline{q})\|_{\mathcal{L}}^{1/2} \leqslant (-1 + \|K_\ell\|_{Z_{Pr}})\|(\theta, \underline{q})\|_{Z_{Pr}}^2 + f_2(t)\|(\theta, \underline{q})\|_{\mathcal{L}}^2 + f_1(t)\|(\theta, \underline{q})\|_{\mathcal{L}} \tag{32}$$

Let $\|K_\ell\|_{Z_{Pr}} < 1$. Because Z_{Pr} and Z have only identical elements, from $Z \subset \mathcal{L}$ follows that $Z_{Pr} \subset \mathcal{L}$. Hence, there exists a positive constant C_1^2 such that

$$(-1 + \|K_\ell\|)\|(\theta, \underline{q})\|^2_{Z_{Pr}} \leqslant C_1^2 \|(\theta, \underline{q})\|^2_{\mathcal{L}} \tag{33}$$

Substitution of (33) in (32) yields

$$\frac{d}{dt} \|(\theta, \underline{q})\|_{\mathcal{L}} \leqslant (-C_1^2 + f_2^2(t))\|(\theta, \underline{q})\|_{\mathcal{L}} + f_1(t) \tag{34}$$

4. Let ϵ satisfy $0 < \epsilon < C_1^2$. Because both $f_1(t)$ and $f_2(t)$ approach zero as $t \to \infty$, there exists a time t' and a positive constant C such that for $t > t'$.

$$\left\{ \begin{aligned} & -C_1^2 + f_2^2(t) \leqslant - C_1^2 + \epsilon^2 = - C^2 < 0 \\ & f_1(t) < \epsilon^2 \end{aligned} \right. \tag{35}$$

Substitution of (35) in (34) yields

$$\frac{d}{dt} \|(\theta, \underline{q})\|_{\mathcal{L}} \leqslant - C^2 \|(\theta, \underline{q})\|_{\mathcal{L}} + \epsilon^2$$

and by integration

$$0 \leqslant \|(\theta, \underline{q})_{(t)}\|_{\mathcal{L}} \leqslant \left\{ \frac{\epsilon^2}{C^2} \left[\exp(-C^2 Z) - \exp(-C^2 t)\right] + \|(\theta, \underline{q})_{(t)}\|_{\mathcal{L}} \exp(C^2 Z) \right\} \exp(-C^2 t)$$

Hence, $(\theta, \underline{q})$ tends to zero in the norm as $t \to \infty$; i.e.

$$\lim_{t\to\infty} \|(\theta, \underline{q})_{(t)}\|_{\mathcal{L}} = 0$$

The physical interpretation of this theorem is direct:
When $\underline{G}_\infty$ and $\nabla \overset{o}{T}_\infty$ are such that $\|K_\ell\|_{Z_{Pr}} < 1$ any internal flow damps out, regardless of the history of the asymptotical values of $\underline{G}_\infty$ and $\nabla \overset{o}{T}_\infty$ *). In other words, the rest state is stable when $\|K_\ell\|_{Z_{Pr}} < 1$. The inequality $\|K_\ell\|_{Z_{Pr}} < 1$ is called the stability criterion.

The computation of the norm of K_ℓ is a numerical problem and approximation methods such as the Ritz'Method and the Weinstein's Method are available.

Theorem II: The solution $(\theta, \underline{q})$ of Eq. (5) cannot attain any small time-independent asymptotical value different from zero unless $\lambda = 1$ is an eigenvalue of K_ℓ.

Proof: Let $(\theta_\infty, \underline{q}_\infty)$, be the asymptotical values of some solution $(\theta, \underline{q})$ of Eq. 5). Hence $(\theta_\infty, \underline{q}_\infty)$ satisfies Eq. 6. Now consider the scalar product of the momentum equation and some $\underline{\phi}$, $(\varphi, \underline{\phi}) \in Z_{Pr}$, and the product of the energy equation and φ. Integration over R leads to:

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*) Note, however, that the internal flow becomes damped as soon as $\underline{G} \times \nabla \overset{o}{T} \equiv 0$ and the operator K_ℓ, defined on $\underline{G}$ and $\nabla \overset{o}{T}$ satisfies $\|K_\ell\|_{Z_{Pr}} < 1$.

$$\begin{cases} \int_R \underline{\phi} \cdot [(\underline{q}_\infty \cdot \nabla)\underline{q}_\infty] dV - Pr^{1/2} \int_R \underline{\phi} \cdot \Delta \underline{q}_\infty dV + \int_R \underline{\phi} \cdot (\theta_\infty \underline{G}_\infty) dV = \int_R \underline{\phi} \cdot \nabla \rho_\infty dV \\ \int_R \varphi \underline{q}_\infty \cdot \nabla \theta_\infty dV - Pr^{-1/2} \int_R \varphi \Delta \theta_\infty dV + \int_R \varphi \underline{q} \cdot \nabla \overset{o}{T}_\infty dV = 0 \end{cases}$$

for all $(\varphi, \underline{\phi}) \in Z_{Pr}$

Because $(\varphi, \underline{\phi}) \in Z_{Pr}$, the use of Green's Theorem

$$\begin{cases} \int_R \underline{\phi} \cdot \Delta \underline{q}_\infty dV = - \int \frac{\partial \phi_i}{\partial x_j} \frac{\partial \phi_i}{\partial x_j} dV \\ \int_R \varphi \Delta \theta \, dV = - \int \frac{\partial \phi}{\partial x_j} \frac{\partial \theta}{\partial x_j} dV \end{cases}$$

and the definitions of K_s, K_ℓ and I lead to

$$-<(\varphi, \underline{\phi}) ; K_s(\theta_\infty, \underline{q}_\infty)> - <(\varphi, \underline{\phi}); K_\ell(\theta_\infty, \underline{q}_\infty) + <(\varphi, \underline{\phi}) ; I(\theta_\infty, \underline{q}_\infty)> = 0$$

for all $(\varphi, \underline{\phi}) \in Z_{Pr}$

or,

$$<(\varphi, \underline{\phi}) ; (K - I)(\theta_\infty, \underline{q}_\infty)> = 0 \tag{36}$$

for all $(\varphi, \phi) \in Z_{Pr}$

Because the element $(\varphi, \underline{\phi})$ is arbitrary Eq.(36) is satisfied only if

$$K(\theta_\infty, \underline{q}_\infty) = I(\theta_\infty, \underline{q}_\infty) \tag{37}$$

Let now $\lambda = 1$ be a regular point of K_ℓ (hence, not an eigenvalue). If some neighborhood of Q, the pair $(\theta_\infty, \underline{q}_\infty) = Q$ is the unique solution of Eq.(37) (see theorem in Preliminary Results)

The operator K_ℓ is completely continuous and its spectrum is discrete. Therefore, even though $\lambda = 1$ may be an eigenvalue of K_ℓ (i.e, if $\|K_\ell\|_{Z_{Pr}} \geqslant 1$) the associated solution does not depend continuously on the physical parameters of the problem and, consequently, is physically inadmissible.

Theorem II implies then that even when $\|K_\ell\|_{Z_{Pr}} \geqslant 1$ the internal flow cannot approach any small time independent asymptotical value, different from zero. In the general case investigated here if was not proved that the rest state is unstable if $\|K_\ell\|_{Z_{Pr}} \geqslant 1$.

However, if $\underline{G}$ is restricted such that it satisfies Eq.(4), the rest state is unstable when $\|K_\ell\|_{Z_{Pr}} \geqslant 1$:

Proof: When $\underline{G}_\infty \sim \nabla \overset{o}{T}_\infty$ (see note on pg. 5) the bilinear functional which defines K_ℓ is symmetric. In this case the associated operator K_ℓ is symmetric. Because K_ℓ is symmetric and continuous it has an eigenvalue λ^+ such that $|\lambda^+| = \|K_\ell\|$. This eigenvalue can be made positive.

Let $\|K_\ell\| = 1 + \epsilon^2$ and let $(\theta^+, \underline{q}^+)$ be the eigenelement associated with $\lambda^+ = 1 + \epsilon^2$; i.e,

$$K_\ell(\theta^+, \underline{q}^+) = (1 + \epsilon^2) I(\theta^+, \underline{q}^+)$$

or

$$<(\varphi,\ \underline{\phi});\ K_\ell(\theta^+,\ \underline{q}^+)> = (1+\epsilon^2) < (\varphi,\ \underline{\phi});\ (\theta^+,\ \underline{q}^+)> \tag{36}$$

for every $(\varphi,\ \underline{\phi}) \in Z_{Pr}$.

By definition Eq.(36) implies

$$- \int_R \theta^+ G_{i\infty} \phi_i \, dV - \int_R \varphi q_i \frac{\partial \overset{o}{T}_\infty}{\partial x_i} dV =$$

$$= (1+\epsilon^2) \int_R \left(Pr^{-1/2} \frac{\partial \theta^+}{\partial x_k} \frac{\partial \varphi}{\partial x_k} + Pr^{1/2} \frac{\partial q_i^+}{\partial x_j} \frac{\partial \phi_i}{\partial x_j} \right) dV \tag{37}$$

$\underline{G}$ and $\nabla \overset{o}{T}$ approach $\underline{G}_{\infty_0}$ and $\nabla_0 \overset{o}{T}_\infty$ asymptotically. Hence all integrals which contain $\underline{G} - \underline{G}_\infty$ and $\nabla T - \nabla \overset{o}{T}_\infty$, in equations (8), (9) and (10), approach zero as $t \to \infty$.

Suppose that after some time the rest state is attained and let $\epsilon(\theta^+,\ \underline{q}^+)$ and $\theta \leqslant \epsilon \|(\theta^+,\ q^+)\| \leqslant 1$, be a mechanical perturbation. After a short time the disturbance in the fluid, $(\theta_d,\ \underline{q}_d)$, satisfies the asymptotic from of Eq.(10); i.e.

$$\frac{d}{dt} \int_R (\theta_d^2 + q_{i_d} q_{i_d}) dV \cong - \epsilon^2 \int_R \left(Pr^{-1/2} \frac{\partial \theta^+}{\partial x_i} \frac{\partial \theta^+}{\partial x_i} + Pr^{1/2} \frac{\partial q_i^+}{\partial x_j} \frac{\partial q_i^+}{\partial x_j} \right) dV +$$

$$+ \int_R \theta^+ \left(G_{i\infty} + \frac{\partial \overset{o}{T}_\infty}{\partial x_i} \right) q_i^+ \, dV$$

Substitution of Eq.(37) in this equation yields

$$\frac{d}{dt} \int_R (\theta^2 + q_{i_d} q_{i_d}) dV \cong \epsilon^4 \int_R \left(Pr^{-1/2} \frac{\partial \theta^+}{\partial x_k} \frac{\partial \theta^+}{\partial x_k} + Pr^{1/2} \frac{\partial q_i^+}{\partial x_j} \frac{\partial q_i^+}{\partial x_j} \right) dV > 0, \tag{38}$$

$\epsilon \|(\theta^+,\ \underline{q}^+)\|_{Z_{Pr}} \ll 1$; all terms which are of the third power in ϵ, in the equation (37), may be neglected. Hence as long as $(\theta_d,\ \underline{q}_d)$ are close to $\epsilon(\theta^+,\ \underline{q}^+)$:

$$\frac{1}{4} \frac{d^2}{dt^2} \int_R (\theta_d^2 + q_{id}\ q_{id}) dV \cong \int_R \left[\left(\frac{\partial \theta}{\partial t} \right)^2 + \frac{\partial q_i}{\partial t} \frac{\partial q_i}{\partial t} \right] dV > 0 \tag{39}$$

From Eq.(38) and Eq.(39) follows that $(\theta_d,\ \underline{q}_d)$ can decay only when it is no longer small; hence, the rest state is not stable.

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